On the size of pairwise-colliding permutations

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***Abstract*—A structured code that improves the previously best known exponential asymptotic lower bound for the maximum cardinality of a pairwise-colliding set of permutations is presented. The main contribution is an explicit construction of an infinite recursion of pairwise-colliding sets of partial-permutations.**

***Index Terms*—graph capacity, permutation codes, extremal combinatorics**

I. Introduction

Let $\mathcal{S}_n$ denote the set of permutations of the numbers $\{1, \ldots, n\}$. Two permutations $\pi, \sigma \in \mathcal{S}_n$ are said to *collide* if there exists a position $i, 1 \le i \le n$, such that $|\pi(i) - \sigma(i)| = 1$. For a fixed $n \ge 2$, a set $\mathcal{T} \subset \mathcal{S}_n$ is said to be *pairwise-colliding* if every two distinct permutations $\sigma, \pi \in \mathcal{T}$ collide. In this article we study the following combinatorial problem that was initially considered by Körner and Malvenuto [4]: Determine

$$T(n) := \max\{|\mathcal{T}| : \mathcal{T} \text{ is pairwise-colliding, } \mathcal{T} \subset \mathcal{S}_n\}.$$

In the next section we will show how the problem naturally arose from studying (zero-error) graph capacities and permutation codes on them.

A. Background and Motivation

Let G be a (possibly countably infinite) graph whose vertex and edge sets are respectively denoted by $V(G)$ and $E(G)$. For positive integer n, the *power graph* G^n is defined to be the graph with vertex set $V(G)^n$ such that two strings $\mathbf{x}, \mathbf{y} \in V(G)^n$ are adjacent in G^n if and only if there exists $i \in \{1, \ldots, n\}$ such that $\{\mathbf{x}_i, \mathbf{y}_i\} \in E(G)$.

One can interpret adjacency in G as distinguishability between symbols in $V(G)$. Then adjacency in the power graph G^n is distinguishability of strings of length n. Clearly the maximum number of n-letter messages which can be sent such that they can be decoded with zero probability of error equals the maximum cardinality of a clique in G^n. This leads to the classical notion of *graph capacity* first studied by Shannon [6]. Note that, in Shannon's equivalent formulation, he considered the complement graph where the lack of an edge represented distinguishability and hence he was interested in the size of the maximum independent set.

Permutation codes are a family of constrained codes that have been increasingly studied due to their importance in flash memories. Körner and Malvenuto [4] defined the notion of *permutation capacity* of a graph combining the notion of permutation codes and graph capacities. Let G be an underlying (usually infinite) graph.

For $A \subset V(G)$, let $\rho(G, A)$ be the maximum cardinality of a clique in the subgraph of $G^{|A|}$ (the power graph) induced by the set consisting of all strings using each element of A exactly once. In other words, the vertices of the subgraph can be indexed by permutations of the elements of A and a maximum cardinality clique corresponds to a zero-error permutation code of largest size obtained using the elements of A. For $n \ge 2$, let

$$\rho(G, n) := \max_{A \subset V(G), |A| = n} \rho(G, A).$$

The *permutation capacity* of the underlying G is defined by

$$\omega(G) := \limsup_{n \to \infty} \frac{1}{n} \log_2 \rho(G, n).$$

The following lemma is immediate.

Lemma 1. *If $B \subset A \subset V(G)$ then $\rho(G, B) \le \rho(G, A)$.*

The proof follows from the following observation: If one appends a fixed permutation of the vertices in $A \setminus B$ to a clique in the subgraph of $G^{|B|}$ induced by B; we induce a clique of the same size in the subgraph of $G^{|A|}$ induced by A.

Consider the infinite line-graph on the integers, i.e. $V(L) = \mathbb{Z}$ and an edge between integers i, j *if and only if* $|i - j| = 1$. Note that for any $A \subset V(L)$ of cardinality n, the subgraph induced by A is isomorphic to a subgraph of the subgraph induced by $\{1, \ldots, n\}$. Hence, from Lemma 1, we have $\rho(L, n) = \rho(L, \{1, \ldots, n\})$. Now it is immediate that $\rho(L, n) = T(n)$, and this is the motivation behind the quantity studied by Körner and Malvenuto.

The main contribution of this article is an improved lower bound on $\omega(L)$. While the numerical improvement is minor over the previously best known bound, we believe that the construction that is used to attain the improvement is rather different from the existing constructions of pairwise-colliding sets and could be used for further improvements.

B. Previous Work: Bounds on $T(n)$ and $\omega(L)$

The following argument (reproduced here) yields an upper bound to $T(n)$ and consequently $\omega(L)$.

Proposition 1 (Proposition 4.2 of [4])**.** $T(n) \le \binom{n}{\lfloor \frac{n}{2} \rfloor}$

Proof. Take any pairwise-colliding set $\mathcal{T}$ of permutations on $\{1,\ldots,n\}$ and let Ψ map $\pi \in \mathcal{T}$ to the binary string

$$(\pi(1) \mod 2, \pi(2) \mod 2, \ldots, \pi(n) \mod 2) \in \{0,1\}^n.$$

Any two distinct permutations from $\mathcal{T}$ collide (i.e. in particular there is a location where the parities are different) and hence must map to different strings, i.e. Ψ is an injection. Further, any permutation maps to a string with exactly $\lfloor \frac{n}{2} \rfloor$ ones. That there are $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ such strings implies the upper bound. ☐

An immediate corollary of the upper bound on $T(n)$ is that

$$\begin{aligned}\omega(L) &= \limsup_{n\to\infty} \frac{1}{n} \log_2 \rho(L,n) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log_2 T(n) \leq 1.\end{aligned}$$

It is conjectured by Körner and Malvenuto [4] that the upper bound on $T(n)$ can be achieved, that is,

Conjecture 1. $T(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Remark 1. This conjecture has been verified until $n = 9$ and for $n = 10$ it has been shown that $251 \leq T(10) \leq 252$ [2]. The lower bound for $T(10)$ (and the constructions for $T(8)$ and $T(9)$) came from an explicit colliding set obtained using a supercomputer via numerical search.

The above conjecture implies a weaker asymptotic conjecture:

Conjecture 2. $\omega(L) = 1$.

We will now provide some (constructive) lower bounds on $T(n)$ and $\omega(L)$. To explain the constructions, we need to define a *pairwise-colliding set of partial-permutations*. A partial-permutation of $\{1,\ldots,n\}$ is defined as a vector (of length n) consisting of a subset S from $\{1,\ldots,n\}$ and a generic (repeatable) symbol $*$ such that the $*$'s can be replaced by elements from $\{1,\ldots,n\} \setminus S$ to form a permutation of $\{1,\ldots,n\}$. We call the elements in S to be the set of *revealed entries* of the partial-permutation. For instance, $(1 * *)$ is a partial-permutation of $\mathcal{S}_3$ since we can replace the $*$'s to form either of the permutations (1 2 3) or (1 3 2); and 1 is the revealed entry.

A set $\mathcal{P}$ consisting of partial-permutations of $\{1,\ldots,n\}$ is said to be *pairwise-colliding* if for every distinct pair of partial-permutations there exists a position $i, 1 \leq i \leq n$, such that $\pi(i)$ and $\sigma(i)$ are revealed entries and $|\pi(i) - \sigma(i)| = 1$.

The canonical construction of valid codes is to consider a *pairwise-colliding set of partial-permutations*, and then recursively replace the $*$'s in the partial-permutations by *pairwise-colliding set of permutations* and hence obtain a recursion. The key is to find an initial *pairwise-colliding set of partial-permutations* which leads to lower bounds on $T(n)$ and $\omega(L)$.

The constructions proposed in literature have been to exhibit a small finite *pairwise-colliding set of partial-permutations*, where the pairwise-collision can be checked by inspection. In contrast we devise a technique of systematically obtaining an arbitrary-sized *pairwise-colliding set of partial-permutations*. Using this technique we are able to modify some existing constructions to improve the lower bound on $\omega(L)$.

1) Lower Bounds on $T(n)$ and $\omega(L)$: We illustrate some previous constructions of *pairwise-colliding set of partial-permutations* and the implications on $T(n)$ and $\omega(L)$ below. We already know that $T(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for $n \leq 9$ as stated earlier.

Construction A [4]: An elementary example of a recursion is to consider the following set of size two

$$\begin{matrix} 1 & * & * & \cdots & * \\ 2 & * & * & \cdots & * \end{matrix}$$

We replace the $*$'s in the first row by a colliding set from $\{2,\ldots,n\}$ and the $*$'s in the second row by a colliding set from $\{1,3,\ldots,n\}$, to form a new subset $\mathcal{T}$ of permutations of $\{1,\ldots,n\}$. Note that $\mathcal{T}$ is pairwise-colliding. By induction, the $*$'s in the first row can be replaced by a set of size $T(n-1)$. Since the induced sub-graph by $\{1,3,\ldots,n\}$ has a sub-graph isomorphic to $\{1,\ldots,n-2\}$, the $*$'s in the second row can be replaced by a colliding set of size $T(n-2)$. Hence we get that

$$T(n) \geq T(n-1) + T(n-2), \quad \forall n \geq 2.$$

This yields $\omega(L) \geq \log_2 \frac{1+\sqrt{5}}{2} \geq 0.694$.

Construction B: The following set of size 3 yields a larger colliding set. Consider a set of partial-permutations of the form

$$\begin{matrix} 1 & 2 & * & * & \cdots & * \\ 2 & * & 1 & * & \cdots & * \\ * & 1 & 2 & * & \cdots & * \end{matrix}$$

The $*$'s in each of the three rows can be replaced by a colliding set from $\{3,\ldots,n\}$, yielding a pairwise-colliding set of $\{1,\ldots,n\}$. Note that, this yields

$$T(n) \geq 3T(n-2), \quad \forall n \geq 3$$

and hence $\omega(L) \geq \log_2 \sqrt{3} \geq 0.792$.

Construction C [1]: Another example appearing in Brightwell et al. [1] is to consider the set of partial-permutations obtained by taking all cyclic shifts of the two sequences

$$\begin{matrix} 1 & 3 & 4 & 2 & * & * & * \\ 3 & 5 & 2 & 1 & 4 & * & * \end{matrix}$$

It can be verified that the resultant 14-element set of partial-permutations is pairwise-colliding. This implies that

$$T(n) \geq 7T(n-4) + 7T(n-5), \quad \forall n \geq 5$$

and hence $\omega(L) \geq \log_2 x \geq 0.8599$, where x is the unique root of $x^5 - 7x - 7 = 0$ in $[1,2]$.

Construction D [1]: This is the best lower bound for $\omega(L)$ known prior to this paper. The construction was obtained using a computer search by Brightwell et al. (Proposition 19 of [1]).

They obtained the following 17-element pairwise-colliding set of partial-permutations

$$\begin{array}{ccccccccccc}
5 & 2 & 3 & 1 & 4 & \ast & \ast & \ast & \ast & \cdots & \ast \\
5 & \ast & 2 & 3 & 1 & 4 & \ast & \ast & \ast & \cdots & \ast \\
5 & 4 & \ast & 2 & 3 & 1 & \ast & \ast & \ast & \cdots & \ast \\
5 & 1 & 4 & \ast & 2 & 3 & \ast & \ast & \ast & \cdots & \ast \\
5 & 3 & 1 & 4 & \ast & 2 & \ast & \ast & \ast & \cdots & \ast \\
5 & 3 & 2 & 4 & 1 & \ast & \ast & \ast & \ast & \cdots & \ast \\
5 & \ast & 3 & 2 & 4 & 1 & \ast & \ast & \ast & \cdots & \ast \\
5 & 1 & \ast & 3 & 2 & 4 & \ast & \ast & \ast & \cdots & \ast \\
5 & 4 & 1 & \ast & 3 & 2 & \ast & \ast & \ast & \cdots & \ast \\
\ast & 2 & 4 & 1 & \ast & 3 & \ast & \ast & \ast & \cdots & \ast \\
4 & \ast & \ast & 2 & 3 & \ast & 1 & \ast & \ast & \cdots & \ast \\
4 & 3 & \ast & \ast & 2 & \ast & 1 & \ast & \ast & \cdots & \ast \\
4 & \ast & \ast & 1 & 3 & 2 & \ast & \ast & \ast & \cdots & \ast \\
4 & 3 & \ast & \ast & \ast & 1 & 2 & \ast & \ast & \cdots & \ast \\
6 & 2 & 3 & \ast & 4 & \ast & 1 & 5 & \ast & \cdots & \ast \\
6 & 4 & 3 & \ast & \ast & 1 & 2 & 5 & \ast & \cdots & \ast \\
6 & 2 & 5 & 1 & \ast & 3 & \ast & 4 & \ast & \cdots & \ast
\end{array}.$$

This set yields the recursion

$$T(n) \geq 5T(n-4) + 9T(n-5) + 3T(n-6), \quad \forall n \geq 8$$

which then implies

$$\omega(L) \geq \log_2 x \geq 0.8604$$

where x is the unique root of $x^6 - 5x^2 - 9x - 3 = 0$ in $[1, 2]$.

II. Main results

We improve the lower bound for $\omega(L)$ by proposing a simpler construction. This construction is based on a slightly different primitive and not by a finite recursion like in previous cases. Our primitive allows us to have an infinite-length recursion for $T(n)$. To explain our approach we start from Construction B and obtain the following improvement.

Proposition 2. $\omega(L) \geq 0.8495$

Notice that this represents a significant improvement over the bound of 0.792 yielded by Construction B. We start from the following colliding set of partial-permutations given by Construction B:

$$\begin{array}{cccccc}
1 & 2 & \ast & \ast & \cdots & \ast \\
2 & \ast & 1 & \ast & \cdots & \ast \\
\ast & 1 & 2 & \ast & \cdots & \ast
\end{array}$$

Replace the largest entry, 2, in the last row by $\ast$ as shown below. Note that the red-colored row still collides with the first row but no longer necessarily collides with the blue-colored row.

$$\begin{array}{cccccc}
1 & 2 & \ast & \ast & \cdots & \ast \\
2 & \ast & 1 & \ast & \cdots & \ast \\
\ast & 1 & \ast & \ast & \cdots & \ast
\end{array}$$

We split the red row into two partial-permutations with locations of one more entry revealed to get a new colliding set of partial-permutations as shown below.

$$\begin{array}{ccccccc}
1 & 2 & \ast & \ast & \ast & \cdots & \ast \\
2 & \ast & 1 & \ast & \ast & \cdots & \ast \\
3 & 1 & \ast & 2 & \ast & \cdots & \ast \\
\ast & 1 & 2 & 3 & \ast & \cdots & \ast
\end{array}$$

If we stop with this colliding set of partial-permutations we will obtain

$$T(n) \geq 2T(n-2) + 2T(n-3), \quad \forall n \geq 4.$$

We will now repeat the entire process of splitting the last partial-permutation into two partial-permutations with locations of one more entry revealed as follows. For ease of exposition we will explain this split using the coloring as done before. First, we recolor the above set as follows:

$$\begin{array}{ccccccc}
1 & 2 & \ast & \ast & \ast & \cdots & \ast \\
2 & \ast & 1 & \ast & \ast & \cdots & \ast \\
3 & 1 & \ast & 2 & \ast & \cdots & \ast \\
\ast & 1 & 2 & 3 & \ast & \cdots & \ast
\end{array}$$

Again observe that the only pair of partial-permutations that does not collide is the red-colored one and the blue-colored one. We will split the red-colored one into two further ones by revealing the positions of their next entry such that both of these collide with the blue-colored one. The split of the last red-colored row by the same procedure as mentioned previosuly yields the colliding set of partial-permutations:

$$\begin{array}{cccccccc}
1 & 2 & \ast & \ast & \ast & \ast & \cdots & \ast \\
2 & \ast & 1 & \ast & \ast & \ast & \cdots & \ast \\
3 & 1 & \ast & 2 & \ast & \ast & \cdots & \ast \\
4 & 1 & 2 & \ast & 3 & \ast & \cdots & \ast \\
\ast & 1 & 2 & 3 & 4 & \ast & \cdots & \ast
\end{array}$$

If we stop with this colliding set of partial-permutations we will obtain

$$T(n) \geq 2T(n-2) + T(n-3) + 2T(n-4), \quad \forall n \geq 5.$$

This process can clearly be continued indefinitely. For illustration, the next split of the last row yields:

$$\begin{array}{ccccccccc}
1 & 2 & \ast & \ast & \ast & \ast & \ast & \cdots & \ast \\
2 & \ast & 1 & \ast & \ast & \ast & \ast & \cdots & \ast \\
3 & 1 & \ast & 2 & \ast & \ast & \ast & \cdots & \ast \\
4 & 1 & 2 & \ast & 3 & \ast & \ast & \cdots & \ast \\
5 & 1 & 2 & 3 & \ast & 4 & \ast & \cdots & \ast \\
\ast & 1 & 2 & 3 & 4 & 5 & \ast & \cdots & \ast
\end{array}$$

Repeating this split of the last row, if we stop at the iteration that has the entries $\{1, \ldots, k\}$ revealed in the last two rows, then we obtain

$$T(n) \geq 2T(n-2) + \sum_{j=3}^{k-1} T(n-j) + 2T(n-k)$$

for all $n \geq k+1$. This construction implies that

$$\omega(L) \geq \log_2(x),$$

where x is the unique root of $x^k - 2x^{k-2} - \sum_{j=3}^{k-1} x^{k-j} - 2 = 0$ in $(1, 2]$. Since $k \geq 3$ is arbitrary, letting $k \to \infty$ we obtain that

$$\omega(L) \geq \log_2(x) \geq 0.8495.$$

where x is the unique root of $1 - 2x^{-2} - \frac{x^{-3}}{1-x^{-1}} = 0$ in $(1, 2]$.

The structure of the infinite collection of pairwise-colliding partial-permutations, for illustration, is as follows:

$$\begin{array}{cccccccccc}
1 & 2 & \ast & \ast & \ast & \ast & \ast & \ast & \ast & \cdots \\
2 & \ast & 1 & \ast & \ast & \ast & \ast & \ast & \ast & \cdots \\
3 & 1 & \ast & 2 & \ast & \ast & \ast & \ast & \ast & \cdots \\
4 & 1 & 2 & \ast & 3 & \ast & \ast & \ast & \ast & \cdots \\
5 & 1 & 2 & 3 & \ast & 4 & \ast & \ast & \ast & \cdots \\
6 & 1 & 2 & 3 & 4 & \ast & 5 & \ast & \ast & \cdots \\
7 & 1 & 2 & 3 & 4 & 5 & \ast & 6 & \ast & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Theorem 1. *It holds that*

$$T(n) \geq 5\left(T(n-4) + T(n-5) + \cdots + T(1)\right)$$

for $n \geq 6$, which implies

$$\omega(L) \geq 0.867$$

Proof. Consider the five infinite matrices listed below and denote them as $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$.

$$\left.\begin{array}{cccccccccc}
1 & 2 & 4 & 3 & \ast & \ast & \ast & \ast & \ast & \cdots \\
1 & 2 & 5 & \ast & 3 & 4 & \ast & \ast & \ast & \cdots \\
1 & 2 & 6 & 4 & 3 & \ast & 5 & \ast & \ast & \cdots \\
1 & 2 & 7 & 4 & 3 & 5 & \ast & 6 & \ast & \cdots \\
1 & 2 & 8 & 4 & 3 & 5 & 6 & \ast & 7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right\} \mathcal{M}_0$$

$$\left.\begin{array}{cccccccccc}
\ast & 1 & 2 & 4 & 3 & \ast & \ast & \ast & \ast & \cdots \\
3 & 1 & 2 & 5 & \ast & 4 & \ast & \ast & \ast & \cdots \\
3 & 1 & 2 & 6 & 4 & \ast & 5 & \ast & \ast & \cdots \\
3 & 1 & 2 & 7 & 4 & 5 & \ast & 6 & \ast & \cdots \\
3 & 1 & 2 & 8 & 4 & 5 & 6 & \ast & 7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right\} \mathcal{M}_1$$

$$\left.\begin{array}{cccccccccc}
3 & \ast & 1 & 2 & 4 & \ast & \ast & \ast & \ast & \cdots \\
\ast & 3 & 1 & 2 & 5 & 4 & \ast & \ast & \ast & \cdots \\
4 & 3 & 1 & 2 & 6 & \ast & 5 & \ast & \ast & \cdots \\
4 & 3 & 1 & 2 & 7 & 5 & \ast & 6 & \ast & \cdots \\
4 & 3 & 1 & 2 & 8 & 5 & 6 & \ast & 7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right\} \mathcal{M}_2$$

$$\left.\begin{array}{cccccccccc}
4 & 3 & \ast & 1 & 2 & \ast & \ast & \ast & \ast & \cdots \\
5 & \ast & 3 & 1 & 2 & 4 & \ast & \ast & \ast & \cdots \\
6 & 4 & 3 & 1 & 2 & \ast & 5 & \ast & \ast & \cdots \\
7 & 4 & 3 & 1 & 2 & 5 & \ast & 6 & \ast & \cdots \\
8 & 4 & 3 & 1 & 2 & 5 & 6 & \ast & 7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right\} \mathcal{M}_3$$

$$\left.\begin{array}{cccccccccc}
2 & 4 & 3 & \ast & 1 & \ast & \ast & \ast & \ast & \cdots \\
2 & 5 & \ast & 3 & 1 & 4 & \ast & \ast & \ast & \cdots \\
2 & 6 & 4 & 3 & 1 & \ast & 5 & \ast & \ast & \cdots \\
2 & 7 & 4 & 3 & 1 & 5 & \ast & 6 & \ast & \cdots \\
2 & 8 & 4 & 3 & 1 & 5 & 6 & \ast & 7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right\} \mathcal{M}_4$$

Note that the infinite matrices $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$ are obtained by cyclically rotating the first five columns of $\mathcal{M}_0$.

A permutation in $\mathcal{S}_n$ is said to be *consistent* with a given partial-permutation if the permutation matches all the revealed entries of the partial-permutation. We will construct a set of pairwise-colliding permutations in $\mathcal{S}_n$ by starting with a $5(n-4) \times n$ matrix of partial-permutations obtained by taking the top $n-4$ rows and leftmost n columns of each of the infinite matrices $\mathcal{M}_i$ for $i = 0, \ldots, 4$.

We first show that any permutation in $\mathcal{S}_n$ that is consistent with a partial-permutation in $\mathcal{M}_i$ is colliding with any permutation that is consistent with a partial-permutation in $\mathcal{M}_j$, for $0 \leq i < j \leq 4$. To establish this consider the five three-element sets $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$ of partial-permutations:

$$\left.\begin{array}{ccccccccc}
1 & 2 & 4 & 3 & \ast & \ast & \ast & \cdots & \ast \\
1 & 2 & 5 & \ast & 3 & 4 & \ast & \cdots & \ast \\
1 & 2 & \ast & 4 & 3 & \ast & \ast & \cdots & \ast
\end{array}\right\} \mathcal{N}_0$$

$$\left.\begin{array}{ccccccccc}
\ast & 1 & 2 & 4 & 3 & \ast & \ast & \cdots & \ast \\
3 & 1 & 2 & 5 & \ast & 4 & \ast & \cdots & \ast \\
3 & 1 & 2 & \ast & 4 & \ast & \ast & \cdots & \ast
\end{array}\right\} \mathcal{N}_1$$

$$\left.\begin{array}{ccccccccc}
3 & \ast & 1 & 2 & 4 & \ast & \ast & \cdots & \ast \\
\ast & 3 & 1 & 2 & 5 & 4 & \ast & \cdots & \ast \\
4 & 3 & 1 & 2 & \ast & \ast & \ast & \cdots & \ast
\end{array}\right\} \mathcal{N}_2$$

$$\left.\begin{array}{ccccccccc}
4 & 3 & \ast & 1 & 2 & \ast & \ast & \cdots & \ast \\
5 & \ast & 3 & 1 & 2 & 4 & \ast & \cdots & \ast \\
\ast & 4 & 3 & 1 & 2 & \ast & \ast & \cdots & \ast
\end{array}\right\} \mathcal{N}_3$$

$$\left.\begin{array}{ccccccccc}
2 & 4 & 3 & \ast & 1 & \ast & \ast & \cdots & \ast \\
2 & 5 & \ast & 3 & 1 & 4 & \ast & \cdots & \ast \\
2 & \ast & 4 & 3 & 1 & \ast & \ast & \cdots & \ast
\end{array}\right\} \mathcal{N}_4$$

Note that any permutation in $\mathcal{S}_n$ that is consistent with a partial-permutation in $\mathcal{M}_i$ is also consistent with a partial-permutation in $\mathcal{N}_i$, for $i = 0, \ldots, 4$. One can verify that any partial-permutation in $\mathcal{N}_i$ collides, on the revealed entries, with any partial-permutation in $\mathcal{N}_j$, for $0 \leq i < j \leq 4$. This establishes that any permutation in $\mathcal{S}_n$ that is consistent with a partial-permutation in $\mathcal{M}_i$ collides with any permutation that is consistent with a partial-permutation in $\mathcal{M}_j$, for $0 \leq i < j \leq 4$.

Therefore to complete our construction of the set of pairwise-colliding permutations in $\mathcal{S}_n$ we first focus on constructing a set of pairwise-colliding permutations that are consistent with partial-permutations in $\mathcal{M}_0$.

Now consider the infinite matrix $\mathcal{M}_0$

$$\begin{array}{ccccccccccc}
1 & 2 & 4 & 3 & \ast & \ast & \ast & \ast & \ast & \ast & \cdots \\
1 & 2 & 5 & \ast & 3 & 4 & \ast & \ast & \ast & \ast & \cdots \\
1 & 2 & 6 & 4 & 3 & \ast & 5 & \ast & \ast & \ast & \cdots \\
1 & 2 & 7 & 4 & 3 & 5 & \ast & 6 & \ast & \ast & \cdots \\
1 & 2 & 8 & 4 & 3 & 5 & 6 & \ast & 7 & \ast & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

and let A_n denote the truncated matrix obtained by taking the top $n-4$ rows and leftmost n columns.

For each $i \in \{1, \ldots, n-4\}$ take a set of largest cardinality $\mathcal{T}_i \subset \mathcal{S}_i$ of pairwise-colliding permutations and let $\mathcal{U}_i \subset \mathcal{S}_n$ be the set of permutations obtained by replacing the $\ast$'s (from left to right) by

$$\pi(1) + i + 3,\ \pi(2) + i + 3,\ \ldots,\ \pi(n-3-i) + i + 3$$

in the i-th row of A_n for each $\pi \in \mathcal{T}_i$. Clearly,

- $\mathcal{U}_i$ is pairwise-colliding and is of cardinality $T(n-3-i)$, $1 \leq i \leq n-4$,
- if $\pi \in \mathcal{U}_i$ and $\sigma \in \mathcal{U}_{i+1}$, for $1 \leq i \leq n-5$ then
$$\pi(3) + 1 = i + 4 = \sigma(3)$$
and hence π, σ collide,
- if $\pi \in \mathcal{U}_1$ and $\sigma \in \mathcal{U}_j$ with $j \geq 3$ then
$$\pi(4) + 1 = 4 = \sigma(4)$$
and hence π, σ collide,
- if $\pi \in \mathcal{U}_i$ and $\sigma \in \mathcal{U}_j$ with $j - 2 \geq i > 1$ then
$$\pi(i+4) + 1 = i + 3 = \sigma(i+4)$$
and hence π, σ collide.

Thus $\mathcal{V}_0 := \bigcup_{i=1}^{n-4} \mathcal{U}_i$ is a set of $\sum_{i=1}^{n-4} T(i)$ pairwise-colliding permutations in $\mathcal{S}_n$. Note that each permutation in $\mathcal{V}_0$ is consistent with some partial-permutation in $\mathcal{M}_0$.

Now similarly for $i = 1, \ldots, 4$, from the infinite matrix $\mathcal{M}_i$, we can construct a set $\mathcal{V}_i$ of $\sum_{i=1}^{n-4} T(i)$ pairwise-colliding permutations in $\mathcal{S}_n$ such that each permutation in $\mathcal{V}_i$ is consistent with some partial-permutation in $\mathcal{M}_i$. Taking the union of the $\mathcal{V}_i$'s ($i = 0, \ldots, 4$) gives a set of $5 \sum_{i=1}^{n-4} T(i)$ pairwise-colliding permutations in $\mathcal{S}_n$. Thus we obtain that

$$T(n) \geq 5\left(T(n-4) + T(n-5) + \cdots + T(1)\right)$$

for $n \geq 6$, as desired.

It remains to deduce the lower bound for $\omega(L)$. Define a sequence $C(n)$ by

$$C(n) := T(n)$$

for $1 \leq n \leq 5$ and

$$C(n) := 5\left(C(n-4) + C(n-5) + \cdots + C(1)\right)$$

for $n \geq 6$. By induction, we can see that $T(n) \geq C(n)$ for every $n \geq 1$. From standard recursion analysis we see that

$$\lim_{n \to \infty} C(n)^{1/n} = \alpha$$

where α is the unique root for $x \mapsto x^4 - x^3 - 5$ in $[1, 2]$. This implies that

$$\omega(L) = \limsup_{n \to \infty} \frac{1}{n} \log_2 T(n) \geq \log_2 \alpha \geq 0.867$$

and shows the lower bound.

□

III. Conclusion

We improve on the existing lower bound for $\omega(L)$, the asymptotic exponent for the maximum cardinality $T(n)$ of a pairwise-colliding set of permutations of $\{1, \ldots, n\}$. We used an infinite recursive structure for the study of this problem and it is very likely that there are other similar recursive structures that would lead to further improvements. This method gives rise to a slightly different way of code constructions akin to variable length codes over fixed length codes.

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